15 Functions 2

Defn Function $f: A \to B$ is one-to-one if $f(a) = f(a') \Rightarrow a = a'$.
(Same outputs must come from same inputs.)

Example 1

f: $0 \to 3$, $1 \to 5$, $2 \to 4$, 6 — one-to-one

g: $0 \to 4$, $1 \to 4$, $2 \to 5$, $3 \to 6$ — not one-to-one (4 is output to 0 & 1)

$h: \mathbb{Z} \to \mathbb{Z}$, $h(x) = 2x$: $\ldots$, $-1 \to -2$, $0 \to 0$, $1 \to 2$, $\ldots$ — one-to-one

$$\boxed{\text{To show } f \text{ is \underline{not} one-to-one, give example where } x_1 \neq x_2 \text{ but } f(x_1) = f(x_2)}$$

Example 2 Show $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \lfloor x \rfloor$ = "round down x to nearest integer $\leq x$" is not one-to-one.

A $2.1 \neq 2.2$ but $f(2.1) = 2 = f(2.2)$ so f is not one-to-one.

$$\boxed{\text{To show } f \text{ is one-to-one, set outputs equal, then try to get inputs equal.}}$$

Example 3 Show $f: \mathbb{R} \to \mathbb{R}$, $f(x) = 2x+1$ is one-to-one.

A $\underbrace{f(x_1) = f(x_2)}_{\text{Same outputs lead to}\ldots} \Rightarrow 2x_1 + 1 = 2x_2 + 1 \xRightarrow{-1} 2x_1 = 2x_2 \xRightarrow{\div 2} \underbrace{x_1 = x_2}_{\ldots\text{same inputs}}$. So f is one-to-one.

Ex 4 Is $g: \mathbb{R} \to \mathbb{R}$, $g(x) = x^2$ one-to-one?

A $g(1) = 1^2 = 1$ and $g(-1) = (-1)^2 = 1$ but $1 \neq -1$ so g is not one-to-one.

Ex 5 Is $h: \mathbb{R}_{>0} \to \mathbb{R}_{>0}$, $h(x) = x^2$ one-to-one? Here $\mathbb{R}_{>0} = \{x \in \mathbb{R} : x > 0\}$.

A $h(x_1) = h(x_2) \Rightarrow (x_1)^2 = (x_2)^2 \xRightarrow{\sqrt{\circ}} x_1 = \pm x_2 \xRightarrow[x_1, x_2 > 0]{} x_1 = x_2$.

So h is one-to-one.

Def'n The <u>composition</u> of $f: A \to B$ and $g: B \to C$ is $g \circ f: A \to C$ defined by

$$(g \circ f)(x) = g(f(x)).$$

Ex 6

$$\begin{array}{ccccc} 0 & \xrightarrow{f(x) = 2x} & 0 & \xrightarrow{g(x) = x^2} & 0 \\ 1 & \longrightarrow & 2 & \longrightarrow & 4 \\ 2 & \longrightarrow & 4 & \longrightarrow & 16 \end{array} \quad \Rightarrow \quad \begin{array}{ccc} 0 & \xrightarrow{g \circ f} & 0 \\ 1 & \longrightarrow & 4 \\ 2 & \longrightarrow & 16. \end{array}$$

Formula for $g \circ f$ and $f \circ g$?

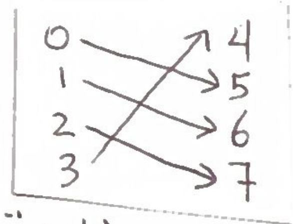
$$(g \circ f)(x) = g(f(x)) = g(2x) = (2x)^2 = 4x^2$$

$$(f \circ g)(x) = f(g(x)) = f(x^2) = 2x^2.$$

Def'n Function $f: A \to B$ is a <u>bijection</u> if f is $\underbrace{\text{one-to-one}}_{\text{each } b \in B \text{ comes from } \leq 1\ a \in A}$ and $\underbrace{\text{onto}}_{\text{each } b \in B \text{ comes from } \geq 1\ a \in A}$.

each $b \in B$ comes from exactly one $a \in A$.

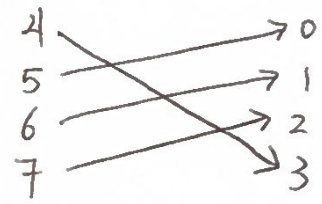
Ex. 7 $f: \{0, 1, 2, 3\} \to \{4, 5, 6, 7\}$



is a bijection

$\xRightarrow{\text{reverse its arrows}}$

$f^{-1}: \{4, 5, 6, 7\} \longrightarrow \{0, 1, 2, 3\}$



"inverse of f"

Def'n The inverse of a bijection $f: A \to B$ is the function $f^{-1}: B \to A$ defined by

$$f^{-1}(b) = a \iff f(a) = b.$$